All-loop finiteness of the two-dimensional noncommutative supersymmetric gauge theory

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Abstract

Within the superfield approach, we discuss the two-dimensional noncommutative super-QED. Its all-order finiteness is explicitly shown.

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Noncommutative field theories have attracted a great deal of interest during the last ten years. One their key characteristic consists in the UV/IR mixing [1] which generates a new type of divergences which can break the perturbative expansion. It is known that the supersymmetry improves the situation, both in the four-dimensional case [2–5] and in the three-dimensional one [6, 7]. A recent paper [8] raised the interest in to two-dimensional noncommutative Yang-Mills theory. Therefore, a natural question is – what is the situation in the supersymmetric extension of this theory? This paper is devoted to this problem.

The action of the two-dimensional noncommutative supersymmetric QED is (following the notations of [9]),

$$S \;=\; \frac{1}{2g^2}\int d^4z W^\alpha * W_\alpha\,, \tag{1}$$

where

$$W_\beta = \frac{1}{2}D^\alpha D_\beta A_\alpha - \frac{i}{2}[A^\alpha, D_\alpha A_\beta] - \frac{1}{6}[A^\alpha, \{A_\alpha, A_\beta\}] \tag{2}$$

is a superfield strength constructed from the spinor superpotential A_α. Hereafter it is implicitly assumed that all commutators and anticommutators are Moyal ones. We note that in the two-dimensional space-time, the noncommutativity matrix is $\Theta^{\mu\nu} \equiv \Theta\epsilon^{\mu\nu}$, where $\epsilon^{\mu\nu}$ is the two-dimensional Levi-Civita symbol. Here we used the fact that the structure of the superfields in two and three dimensions is the same (cf. [9, 10]).

The action in Eq. (1) is invariant under the infinitesimal gauge transformations

$$\delta A_\alpha = D_\alpha K - i[A_\alpha, K]\,. \tag{3}$$

After gauge fixing, the total action of the noncommutative supersymmetric QED reads

$$S_{total} = S + S_{GF} + S_{FP}\,, \tag{4}$$

where S_{GF} is the gauge fixing term,

$$S_{GF} = -\frac{1}{4\xi g^2}\int d^4z (D^\alpha A_\alpha) D^2 (D^\beta A_\beta)\,, \tag{5}$$

and S_{FP} is the corresponding action for the Faddeev-Popov ghosts,

$$S_{FP} = \frac{1}{2g^2}\int d^4z (c' D^\alpha D_\alpha c + ic' * D^\alpha\{A_\alpha, c\})\,. \tag{6}$$

From Eq. (4), one obtains the propagator for the gauge fields,

$$< A^{\alpha}(-p,\theta_1)A^{\beta}(p,\theta_2) >= \frac{1}{i}g^2\left[-\frac{D^2D^{\beta}D^{\alpha}}{2p^4}+\xi\frac{D^2D^{\alpha}D^{\beta}}{2p^4}\right]\delta_{12}\,, \tag{7}$$

and the propagator for the ghost fields,

$$< c'(-k,\theta_1)c(k,\theta_2) >= ig^2\frac{D^2}{k^2}\delta_{12}\,, \tag{8}$$

where, $\delta_{12}\equiv\delta^2(\theta_1-\theta_2)$ is the usual Grassmannian delta function. Finally, the interaction part of the classical action in the pure gauge sector is

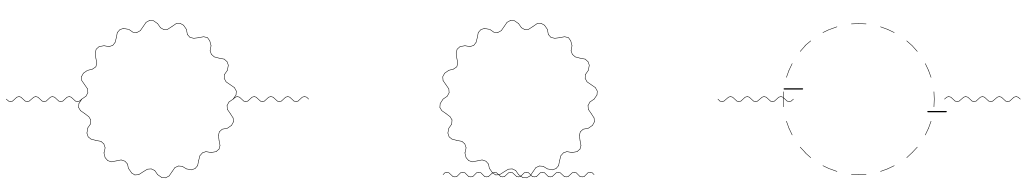
$$\begin{aligned} S_{int} \;=\; & \frac{1}{g^2}\int d^4z\Big[-\frac{i}{4}D^{\gamma}D^{\alpha}A_{\gamma}*[A^{\beta},D_{\beta}A_{\alpha}]-\frac{1}{12}D^{\gamma}D^{\alpha}A_{\gamma}*[A^{\beta},\{A_{\beta},A_{\alpha}\}]- \\ & -\frac{1}{8}[A^{\gamma},D_{\gamma}A^{\alpha}]*[A^{\beta},D_{\beta}A_{\alpha}]+\frac{i}{12}[A^{\gamma},D_{\gamma}A^{\alpha}]*[A^{\beta},\{A_{\beta},A_{\alpha}\}]+ \\ & +\frac{1}{72}[A^{\gamma},\{A_{\gamma},A^{\alpha}\}]*[A^{\beta},\{A_{\beta},A_{\alpha}\}]\Big]\,, \end{aligned} \tag{9}$$

from which the interacting vertices for the perturbative calculations may be read directly.

The possible divergences of the theory are specified by the superficial degree of the divergence (for details see [6])

$$\omega = 1-\frac{1}{2}V_c-2V_0-\frac{3}{2}V_1-V_2-\frac{1}{2}V_3-\frac{1}{2}N_D\,, \tag{10}$$

where V_i is the number of purely gauge vertices involving i supercovariant derivatives, V_c is the number of gauge-ghost vertices, and N_D is the number of spinor derivatives acting on the external fields. It is easy to see that the theory is super-renormalizable, with only (logarithmically) divergent graphs which are those with $V_3=2$, or $V_2=1$, or $V_c=2$. They contribute to the one-loop two-point function of A^{α} field and are depicted in the figure below.



a $\qquad$ b $\qquad$ c

In these graphs, a cut in a ghost line corresponds to the factor D_{α} acting on the ghost propagator. A trigonometric factor $e^{ik\wedge l}-e^{il\wedge k}=2i\sin(k\wedge l)$, where $k\wedge l\equiv k^{\mu}l^{\nu}\Theta\epsilon_{\mu\nu}$, originates from each commutator. By denoting the contributions of the graphs in the figure

above by I_{1a}, I_{1b}, and I_{1c}, respectively, after some D-algebra transformations we arrive at

$$I_{1a} = \frac{1}{2}\xi \int \frac{d^2p}{(2\pi)^2} d^2\theta_1 \int \frac{d^2k}{(2\pi)^2} \frac{\sin^2(k\wedge p)}{k^2} A^\beta(-p,\theta_1)A_\beta(p,\theta_1) + \cdots, \tag{11}$$

$$I_{1b} = \frac{1}{2}(1-\xi) \int \frac{d^2p}{(2\pi)^2} d^2\theta_1 \int \frac{d^2k}{(2\pi)^2} \frac{\sin^2(k\wedge p)}{k^2} A^\beta(-p,\theta_1)A_\beta(p,\theta_1) + \cdots, \tag{12}$$

$$I_{1c} = -\frac{1}{2} \int \frac{d^2p}{(2\pi)^2} d^2\theta_1 \int \frac{d^2k}{(2\pi)^2} \frac{\sin^2(k\wedge p)}{k^2} A^\beta(-p,\theta_1)A_\beta(p,\theta_1) + \cdots. \tag{13}$$

where the elipsis stand for the finite parts. Hence, the total one-loop two-point function of the gauge superfield, given by $I_1 = I_{1a} + I_{1b} + I_{1c}$, is free from both UV and UV/IR infrared singularities.

We already mentioned that divergences are possible only for $V_2 = 1$, or $V_3 = 2$, or $V_c = 2$. It is easy to see that two-loop graphs satisfying these conditions are just vacuum ones whereas higher-loop graphs cannot satisfy these conditions at all. Therefore, there are no UV and UV/IR infrared divergences beyond one-loop and, as a consequence, the theory is finite at any loop order. The generalization for the non-Abelian case (where $A_\alpha(z) = A^a_\alpha(z)T^a$, with T^a being the generators of the gauge group in the fundamental representation) is straightforward, and by repeating the three-dimensional calculations, we have again, as in three and four dimensions, that at

$$\mathrm{tr}(T_aT_bT_aT_c) = 2\mathrm{tr}(T_aT_bT_d)\mathrm{tr}(T_aT_cT_d) \tag{14}$$

all one-loop divergences explicitly cancel, while the higher-loop ones simply do not arise, i.e. the pure two-dimensional gauge theory is completely finite!

We next study the interaction of the spinor gauge field with matter. To this end we add to Eq. (4) the action of the N scalar matter superfields ϕ_a, with $a = 1, \ldots, N$.

$$S_m = \int d^4z \Big[-\bar\phi_a(D^2 - m)\phi_a + i\frac{g}{2}([\bar\phi_a, A^\alpha] * D_\alpha\phi_a - D_\alpha\bar\phi_a * [A^\alpha, \phi_a]) + + \frac{g^2}{2}[\bar\phi_a, A^\alpha] * [A_\alpha, \phi_a]\Big], \tag{15}$$

The free propagator of the scalar superfields is

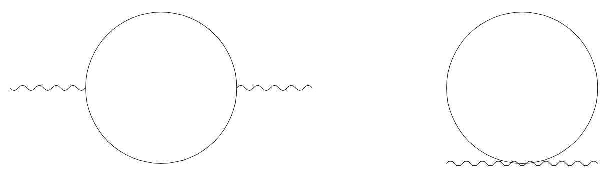
$$<\bar\phi_a(-k,\theta_1)\phi_b(k,\theta_2)> = i\delta_{ab}\frac{D^2+m}{k^2+m^2}\delta_{12}, \tag{16}$$

and the superficial degree of divergence when matter fields are present can be shown to be equal to

$$\omega = 1 - \frac{1}{2}V_c - 2V_0 - \frac{3}{2}V_1 - V_2 - \frac{1}{2}V_3 - \frac{1}{2}E_\phi - \frac{1}{2}V^1_\phi - \frac{1}{2}N_D - V^0_\phi, \tag{17}$$

where, as before, V_i is the number of pure gauge vertices with i spinor derivatives, E_ϕ is the number of external scalar lines, N_D is the number of spinor derivatives associated to external lines, V_ϕ^1 is the number of triple vertices involving matter, and V_ϕ^0 is the number of quartic vertices involving matter. We note that the superficial degree of divergence of any supergraph in the two-dimensional theory is evidently less by one than the superficial degree of divergence of the same supergraph in the three-dimensional theory. It is straightforward to show that the graphs with non-zero number of external matter legs, possessing $E_\phi \geq 2$ together with $V_\phi^1 > 0$ or $V_\phi^0 > 0$, are finite.

It remains to study the graphs with zero number of external matter legs. The leading nontrivial UV divergence for them is presented by the graphs with two external A_α legs which are superficially UV logarithmic divergent. They are depicted in the figure below.



a *b*

Their contribution can be found in the same way as in [6], so here we merely quote the result,

$$\begin{aligned} I_4 &= 2N \int \frac{d^2p}{(2\pi)^2} d^2\theta \int \frac{d^2k}{(2\pi)^2} \frac{\sin^2(k \wedge p)}{(k^2+m^2)\,[(k+p)^2+m^2]} \\ &\times \; (k_{\gamma\beta} - mC_{\gamma\beta}) \Big[(D^2 A^\gamma(-p,\theta)) A^\beta(p,\theta) + \frac{1}{2} D^\gamma D^\alpha A_\alpha(-p,\theta) A^\beta(p,\theta) \Big] \,. \end{aligned} \tag{18}$$

This result is finite which leads to the conclusion that the theory is finite. The non-Abelian case does not essentially differ and again the theory is finite when the matrices for the generators of the gauge group satisfy (14).

We explicitly proved the finiteness of the two-dimensional noncommutative supersymmetric QED. We found that the proof of its finiteness does not essentially differ from the three-dimensional studies [6, 7], and the finiteness is caused by the lower dimension of the space-time. However, in the two-dimensional case, all divergences vanish already in the one-loop order and so the theory is all-loop finite.

Nevertheless, it is interesting to know that there is a strong argument in favor of the two-loop (and hence all-loop) finiteness of the three-dimensional noncommutative supersymmetric QED. Indeed, it follows from [6, 7] that the only potentially divergent structure in two loops looks like $\int d^5 z A^\alpha A_\alpha$. However, the appearance of such a correction is forbidden

by the gauge invariance, which allows only terms with at least two derivatives acting on the external fields. Such a contribution is evidently finite due to (17). By analogy with [11], one can expect that the term $\int d^5 z A^\alpha A_\alpha$ would vanish at least in some specific gauge.

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APPENDIX

Here, we collect some conventions for the two-dimensional supersymmetric field theories used in this work (cf. [9]). We adopt the metric $\eta_{mn} = \mathrm{diag}(-+)$, the Dirac gamma matrices are $(\gamma^0)^\alpha{}_\beta = -i\sigma^2, (\gamma^1)^\alpha{}_\beta = \sigma^1$, where σ^i, $i = 1, 2$ are Pauli matrices, and $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$. To raise and lower spinor indices we use the antisymmetric spinor metric $C_{\alpha\beta} = -C^{\alpha\beta}$, with $C^{12} = i$, following the "north-western" convention, $\psi^\alpha = C^{\alpha\beta}\psi_\beta$ and $\psi_\alpha = \psi^\beta C_{\beta\alpha}$. We define $\psi^2 = \frac{1}{2}\psi^\alpha\psi_\alpha$ and use the identity $A_{[\alpha}B_{\beta]} = -C_{\alpha\beta}A^\gamma B_\gamma$. We employ a bispinor notation so that any vector index is represented by two spinor indices: $A^m \to A^{\alpha\beta} \equiv A^m(\gamma_m)^{\alpha\beta}$. Since the gamma matrices with two lower indices are $(\gamma^m)_{\alpha\beta} = (-\mathbf{1}_2, -\sigma^3)$, all vectors are represented by symmetric bispinors. As a result, the superfields can be defined just as in [9].

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